

Even and odd generalized hypergeometric coherent states

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Abstract

In this paper, we investigate a large class of generalized hypergeometric states $|p, q, z\rangle$, depending on a complex variable z and two sets of parameters, (a_1, \dots, a_p) and (b_1, \dots, b_q) . Even and odd generalized hypergeometric states $|p, q, z\rangle_e$ and $|p, q, z\rangle_o$ are also defined and analyzed. The moment problem is solved by the Mellin transform techniques. For particular values of p and q , the photon-counting statistics, quantum optical properties and geometry of these states are discussed.

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I. INTRODUCTION

Coherent states are at the heart of important investigations in physics and mathematical physics since their potential applications were highlighted in [1], specifically in quantum optics [2]. They were introduced by Schrödinger [3], but also studied by Glauber [4] as the eigenstates of the boson annihilation operator \hat{a} , i.e.,

$$\hat{a}|z\rangle = z|z\rangle \quad (1)$$

where $[\hat{a}, \hat{a}^\dagger] = I$, z is a complex number with conjugate \bar{z} ; $|z\rangle$ are the normalized states given by

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle, \quad (2)$$

where $|n\rangle$ is an element of the Fock space $\equiv \{|n\rangle, n = 0, 1, \dots\}$. The coherent states based on the Heisenberg-Weyl group were extended for a number of Lie groups with square integrable representations. For more details on their applications in physics, see also [5–8].

Later on, Klauder et al [9] exposed a general method for constructing holomorphic coherent states of the form

$$|z\rangle = \frac{1}{\sqrt{N(|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho(n)}} |n\rangle \quad (3)$$

where z is a complex variable, $\rho(n)$ is a set of strictly positive parameters and the states $|n\rangle$ form an orthonormal basis. The normalization function is given by

$$N(x) = \sum_{n=0}^{\infty} \frac{x^n}{\rho(n)}, \quad x = |z|^2. \quad (4)$$

Its radius of convergence determines the domain of definition of the states in (3).

The idea of the even and odd coherent states was first introduced by Dodonov, Malkin and Man'ko [10]. Nieto and Traux [11] showed that these states are a special set of nonclassical states. Their properties were studied by some authors [12, 13]. The even coherent states look like squeezed vacuum states [14], because they are the superposition of the photon number states with an even number of quanta. Thus, the even coherent light can be used in interferometric gravitational wave detectors to give the same effort of increasing the sensitivity of these devices. The photon statistics of one-mode even and odd coherent states possesses the nonclassical property of light.

The even and odd coherent states were generalized to the case of even and odd nonlinear coherent states [15, 16] defined as the eigenstates of the operator $f(N)\hat{a}^2$. In this case, the

superposition of even number of states yields the even nonlinear coherent states, while the superposition of odd number of states determines the odd nonlinear coherent states. For $f(N) = 1$, the even nonlinear coherent state becomes the even coherent state and the odd nonlinear coherent state gives the odd coherent state. Depending on the expression of $f(N)$, the even and odd nonlinear coherent states may exhibit various nonclassical features. Thus, the squeezed vacuum state and the squeezed first excited state of the harmonic oscillator can be interpreted as even and odd nonlinear coherent states, with $f(N) = 1/(1 + N)$ and $f(N) = 1/(2 + N)$, respectively.

In this paper, we investigate a large class of generalized hypergeometric states $|p, q, z\rangle$, depending on a complex variable z and two sets of parameters, (a_1, \dots, a_p) and (b_1, \dots, b_q) . Besides, we define and analyze even and odd generalized hypergeometric states $|p, q, z\rangle_e$ and $|p, q, z\rangle_o$. We solve the moment problem using the Mellin transform techniques. For particular values of p and q , we discuss the photon-counting statistics, quantum optical properties and geometry of these states.

The paper is organized as follows. In Section II, we recall known generalized hypergeometric states, define and analyze the even and odd generalized hypergeometric states. In Section III, we discuss their overcompleteness properties. The associated moment problem is solved. Photon-counting statistics is discussed in the Section IV. We end with the study of quantum optical and thermodynamical properties and geometry of these states for particular relevant values of p, q in Section V.

II. EVEN AND ODD GENERALIZED HYPERGEOMETRIC STATES

Let us consider the Fock space $\mathcal{F} := \{|n\rangle, \quad n = 0, 1, 2, \dots\}$ with the orthogonality and completeness conditions

$$\langle n|m\rangle = \delta_{n,m} \quad \text{and} \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = I \quad (5)$$

such that the state $|n\rangle$ is obtained by acting the creation operator \hat{a}^\dagger of boson algebra on the vacuum $|0\rangle$ repeatedly as follows:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle. \quad (6)$$

The actions of the operators \hat{a} and \hat{a}^\dagger on the state $|n\rangle$ are given by

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \text{and} \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \quad (7)$$

As introduced in [17], the generalized hypergeometric states (GHS) are defined as

$$|p, q, z\rangle = |a_1, \dots, a_p; b_1, \dots, b_q; z\rangle = N(|z|^2; p, q) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{{}_p\rho_q(n)}} |n\rangle, \quad (8)$$

where the normalization function $N(x; p, q)$ is afforded by the generalized hypergeometric function:

$$N(x; p, q) = [{}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x)]^{-1/2}, \quad (9)$$

with

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n \cdots (b_q)_n n!} \quad (10)$$

and the strictly positive parameter function

$${}_p\rho_q(n) = n! \frac{(b_1)_n \cdots (b_q)_n}{(a_1)_n \cdots (a_p)_n}, \quad (11)$$

where the Pochhammer-symbol $(a)_n$ is defined as [18]

$$(a)_n = a(a+1) \cdots (a+n-1), \quad n \geq 1, \quad (a)_0 = 1.$$

The function ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x)$ converges in the following cases: for

$$\text{any } z \text{ if } p < q + 1, \quad (12)$$

$$|z| < 1 \text{ if } p = q + 1, \quad (13)$$

$$|z| = 1 \text{ if } p = q + 1, \quad \eta = 1, \quad (14)$$

$$|z| = 1, \quad z \neq 1 \text{ if } p = q + 1, \quad 0 \leq \eta \leq 1, \quad (15)$$

where

$$\eta = \operatorname{Re} \left(\sum_{j=1}^p a_j - \sum_{j=1}^q b_j \right). \quad (16)$$

In all other cases, it diverges [19]. The GHS depend on the complex variable z and on the two sets of sequences (a_1, \dots, a_p) and (b_1, \dots, b_q) . We suppose that the parameters $a_1, \dots, a_p, b_1, \dots, b_q$ are non zero, non negative real parameters. The parameter functions ${}_p\rho_q(n)$ is a real and strictly positive and

$$\frac{(b_1 + n)(b_2 + n) \cdots (b_q + n)}{(a_1 + n)(a_2 + n) \cdots (a_p + n)} > 0, \quad n = 0, 1, 2, \dots. \quad (17)$$

Let us define the even generalized hypergeometric states (EGHS) and the odd generalized hypergeometric states (OGHS) as follows:

$$|p, q, z\rangle_e := \frac{1}{\sqrt{2}} N'_e(x; p, q) (|p, q, z\rangle + |p, q, -z\rangle)$$

and

$$|p, q, z\rangle_o := \frac{1}{\sqrt{2}} N'_o(x; p, q) (|p, q, z\rangle - |p, q, -z\rangle), \quad (18)$$

respectively, where the normalization functions $N'_e(x; p, q)$ and $N'_o(x; p, q)$ are given by

$$N'_e(x; p, q) = \left[\frac{{}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x)}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \right]^{1/2}$$

and

$$N'_o(x; p, q) = \left[\frac{{}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x)}{{}_pS_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \right]^{1/2}. \quad (19)$$

The even generalized hypergeometric function is defined as

$${}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_{2n} \cdots (a_p)_{2n}}{(b_1)_{2n} \cdots (b_q)_{2n}} \frac{x^{2n}}{(2n)!} \quad (20)$$

while the odd generalized hypergeometric function is provided by

$${}_pS_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_{2n+1} \cdots (a_p)_{2n+1}}{(b_1)_{2n+1} \cdots (b_q)_{2n+1}} \frac{x^{2n+1}}{(2n+1)!}. \quad (21)$$

Therefore, the even and odd generalized hypergeometric function satisfy

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = {}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x) + {}_pS_q(a_1, \dots, a_p; b_1, \dots, b_q; x) \quad (22)$$

and

$$\begin{aligned} & [{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x)]^2 - [{}_pS_q(a_1, \dots, a_p; b_1, \dots, b_q; x)]^2 \\ &= {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; -x). \end{aligned} \quad (23)$$

The EGHS and OGHS can be rewritten as

$$|p, q, z\rangle_e = \frac{1}{\sqrt{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; |z|^2)}} \sum_{n=0}^{\infty} \sqrt{\frac{(a_1)_{2n} \cdots (a_p)_{2n}}{(2n)!(b_1)_{2n} \cdots (b_q)_{2n}}} z^{2n} |2n\rangle \quad (24)$$

and

$$|p, q, z\rangle_o = \frac{1}{\sqrt{{}_pS_q(a_1, \dots, a_p; b_1, \dots, b_q; |z|^2)}} \sum_{n=0}^{\infty} \sqrt{\frac{(a_1)_{2n+1} \cdots (a_p)_{2n+1}}{(2n+1)!(b_1)_{2n+1} \cdots (b_q)_{2n+1}}} z^{2n+1} |2n+1\rangle. \quad (25)$$

From (24) and (25), the EGHS and OGHS satisfy the orthonormality condition

$${}_e\langle z, p, q | p, q, z \rangle_o = 0 \quad (26)$$

and the action of the annihilation operator on them yields, respectively,

$$\hat{a}|p, q, z\rangle_e = z \sqrt{\frac{a_1 \cdots a_p}{b_1 \cdots b_q}} \sqrt{\frac{{}_pS_q(a_1 + 1, \dots, a_p + 1; b_1 + 1, \dots, b_q + 1; |z|^2)}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; |z|^2)}} |p, q, z\rangle_o$$

and

$$\hat{a}|p, q, z\rangle_o = z \sqrt{\frac{a_1 \cdots a_p}{b_1 \cdots b_q}} \sqrt{\frac{{}_pC_q(a_1 + 1, \dots, a_p + 1; b_1 + 1, \dots, b_q + 1; |z|^2)}{{}_pS_q(a_1, \dots, a_p; b_1, \dots, b_q; |z|^2)}} |p, q, z\rangle_e, \quad (27)$$

where the identities

$$(a)_n = a(a+1)_{n-1}, \quad (a)_n = a(a+1)(a+2)_{n-2}$$

are used. The EGHS and OGHS are also continuous in their label z .

III. COMPLETENESS

The overcompleteness property consists of finding a positive function ${}_pW_q(|z|^2)$ such that

$$\frac{1}{\pi} \int \int_D d^2z |p, q, z\rangle {}_pW_q(|z|^2) \langle z, p, q| = I = \sum_{n=0}^{\infty} |n\rangle \langle n|, \quad (28)$$

where D is a disc in the complex plane centered at the origin, of radius R and $d^2z = |z|d|z|d\theta$. By substituting $z = \sqrt{x} e^{i\theta}$ in this equation, and evaluating the integral over θ in the l.h.s of (28), we finally arrive at

$$\int_0^R x^n \left[\frac{{}_pW_q(x)}{N^2(x; p, q)} \right] dx = {}_p\rho_q(n), \quad x = |z|^2. \quad (29)$$

Hence, (29) is then the power moments of the unknown function ${}_pW_q(x)$ and is the Stieltjes moment problem for $R = \infty$ or the Hausdorff moment problem for $R < \infty$. These are classical mathematical problems on which an extensive and mathematically oriented literature exists, see for instance [20–22] and references therein. Under the considerations (12) and (13), the moment problem (29) takes the form

$$\int_0^R x^n {}_p\omega_q(x) dx = {}_p\rho_q(n), \quad (30)$$

where ${}_p\omega_q(x) = {}_pW_q(x)/N^2(x; p, q)$. Its solution can be obtained by using Mellin transform techniques [23, 24]. By replacing n by $s-1$ in (30), the distribution ${}_p\omega_q(x)$ and the parameter function ${}_p\rho_q(s-1)$ become a Mellin transform related pair [19, 26]:

$$\int_0^\infty x^{s-1} {}_p\omega_q(x) dx = \Gamma(s) \frac{\prod_{i=1}^q \Gamma(b_i - 1 + s)}{\prod_{i=1}^p \Gamma(a_i - 1 + s)}, \quad x = |z|^2. \quad (31)$$

For $p < q + 1$ and $p = q + 1$, $\eta > 1$, the distribution ${}_p\omega_q(x)$ is given by [17]

$$\begin{aligned} {}_p\omega_q(x) &= {}_p\omega_q(x; a_1, \dots, a_p; b_1, \dots, b_q) \\ &= \frac{\Gamma(a_1) \cdots \Gamma(a_p)}{\Gamma(b_1) \cdots \Gamma(b_p)} G_{p,q+1}^{q+1,0} \left(x \left| \begin{matrix} a_1 - 1, \dots, a_p - 1 \\ b_1 - 1, \dots, b_q - 1, 0 \end{matrix} \right. \right) \end{aligned} \quad (32)$$

and the weight function ${}_pW_q(x)$ is furnished by the expression

$$\begin{aligned} {}_pW_q(x) &= {}_pW_q(x; a_1, \dots, a_p; b_1, \dots, b_q) \\ &= \frac{\Gamma(a_1) \cdots \Gamma(a_p)}{\Gamma(b_1) \cdots \Gamma(b_p)} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) \\ &\quad \times G_{p,q+1}^{q+1,0} \left(x \left| \begin{matrix} a_1 - 1, \dots, a_p - 1 \\ b_1 - 1, \dots, b_q - 1, 0 \end{matrix} \right. \right) \end{aligned} \quad (33)$$

where G is the Meijer function [19, 26].

For the EGHS and the OGHS, the moment problem (30) takes the form

$$\int_0^\infty x^{2n} {}_p\omega_q^e(x) dx = {}_p\rho_q(2n), \quad (34)$$

and

$$\int_0^\infty x^{2n+1} {}_p\omega_q^o(x) dx = {}_p\rho_q(2n+1), \quad (35)$$

respectively, where ${}_p\omega_q^e(x) = {}_pW_q^e(x)/{}_pC_q(x; p, q)$ and ${}_p\omega_q^o(x) = {}_pW_q^o(x)/{}_pS_q(x; p, q)$.

In particular,

1. For $p = 0 = q$, the states $|0, 0, z\rangle_e = |z\rangle_e$ and $|0, 0, z\rangle_o = |z\rangle_o$ related to ${}_0\rho_0(n) = n!$ have the normalization functions given by [16]

$${}_0C_0(-; -; x) = \cosh x, \quad {}_0S_0(-; -; x) = \sinh x \quad (36)$$

and the corresponding weight functions are

$${}_0\omega_0^e(x) = {}_0\omega_0^e(x; -; -) \equiv {}_0\omega_0^o(x) = {}_0\omega_0^o(x; -; -) = e^{-x}. \quad (37)$$

2. For $p = 0$, $q = 1$, the states $|0, 1, z\rangle_e = |-, b, z\rangle_e$ and $|0, 1, z\rangle_o = |-, b, z\rangle_o$ related to ${}_0\rho_1(n) = n!(b)_n$ for $b > 0$ due to (17) lead to the normalization functions

$${}_0C_1(-; b; x) = {}_0F_3 \left(\begin{matrix} - \\ \frac{1}{2}, \frac{b}{2}, \frac{b+1}{2} \end{matrix} \left| \frac{x^2}{16} \right. \right), \quad {}_0S_1(-; b; x) = \frac{4x}{b} {}_0F_3 \left(\begin{matrix} - \\ \frac{3}{2}, \frac{b+1}{2}, \frac{b+2}{2} \end{matrix} \left| \frac{x^2}{16} \right. \right) \quad (38)$$

and the weight functions by ([27], p 196, formula (5.39)) are reduced to

$${}_0\omega_1^e(x) = {}_0\omega_1^e(x; -, b) \equiv {}_0\omega_1^o(x) = {}_0\omega_1^o(x; -, b) = \frac{2}{\Gamma(b)} x^{\frac{b-1}{2}} K_{b-1}(2\sqrt{x}), \quad (39)$$

where $K_\alpha(\cdot)$ is the modified Bessel function [27].

3. For $p = 1, q = 0$, the states $|1, 0, z\rangle_e = |a, -, z\rangle_e$ and $|1, 0, z\rangle_o = |a, -, z\rangle_o$ originate from ${}_1\rho_0(n) = n!/(a)_n$, with $a > 0$ due to (17). The normalization functions are given by

$${}_1C_0(a; -, x) = {}_2F_1\left(\begin{matrix} \frac{a}{2}, \frac{a+1}{2} \\ \frac{1}{2} \end{matrix} \middle| x^2\right), \quad {}_1S_0(a; -, x) = ax {}_2F_1\left(\begin{matrix} \frac{a+1}{2}, \frac{a+2}{2} \\ \frac{3}{2} \end{matrix} \middle| x^2\right). \quad (40)$$

The corresponding weight functions by ([27], p 195, formula (5.35)) are expressed as follows:

$${}_1\omega_0^e(x) = {}_1\omega_0^e(x; a; -) \equiv {}_1\omega_0^o(x) = {}_1\omega_0^o(x; a; -) = (a-1)(1-x)^{a-2}. \quad (41)$$

4. For $p = 1 = q$, the states $|1, 1, z\rangle_e = |a, b, z\rangle_e$ and $|1, 1, z\rangle_o = |a, b, z\rangle_o$ originate from ${}_1\rho_1(n) = n!(b)_n/(a)_n$, with $a, b > 0$ due to (17). The normalization functions are given by

$${}_1C_1(a; b; x) = {}_2F_3\left(\begin{matrix} \frac{a}{2}, \frac{a+1}{2} \\ \frac{1}{2}, \frac{b}{2}, \frac{b+1}{2} \end{matrix} \middle| \frac{x^2}{4}\right), \quad {}_1S_1(a; b; x) = \frac{2ax}{b} {}_2F_3\left(\begin{matrix} \frac{a+1}{2}, \frac{a+2}{2} \\ \frac{3}{2}, \frac{b+1}{2}, \frac{b+2}{2} \end{matrix} \middle| \frac{x^2}{4}\right). \quad (42)$$

The corresponding weight functions by ([27], p 197, formula (5.46)) can be expressed in terms of the Whittaker function as follows:

$${}_1\omega_1^e(x) = {}_1\omega_1^e(x; a; b) \equiv {}_1\omega_1^o(x) = {}_1\omega_1^o(x; a; b) = \frac{\Gamma(a)}{\Gamma(b)} e^{-\frac{x}{2}} W_{\frac{1+b}{2}-a, -\frac{b}{2}}(x), \quad (43)$$

where $W_{\alpha, \beta}(\cdot)$ is the Whittaker function [27].

5. For $p = 2, q = 1$, the states $|2, 1, z\rangle_e = |a_1, a_2, b, z\rangle_e$ and $|2, 1, z\rangle_o = |a_1, a_2, b, z\rangle_o$ originate from ${}_2\rho_1(n) = n!(b)_n/(a_1, a_2)_n$, with $a_1, a_2, b > 0$ due to (17). The normalization functions are given in terms of hypergeometric function ${}_4F_3$:

$${}_2C_1(a_1, a_2; b; x) = {}_4F_3\left(\begin{matrix} \frac{a_1}{2}, \frac{a_2}{2}, \frac{a_1+1}{2}, \frac{a_2+1}{2} \\ \frac{1}{2}, \frac{b}{2}, \frac{b+1}{2} \end{matrix} \middle| x^2\right), \quad (44)$$

$${}_2S_1(a_1, a_2; b; x) = \frac{a_1 a_2 x}{b} {}_4F_3\left(\begin{matrix} \frac{a_1+1}{2}, \frac{a_2+1}{2}, \frac{a_1+2}{2}, \frac{a_2+2}{2} \\ \frac{3}{2}, \frac{b+1}{2}, \frac{b+2}{2} \end{matrix} \middle| x^2\right) \quad (45)$$

while the corresponding weight functions by ([27], p 198, formula (5.50)) are obtained in terms of ${}_2F_1$:

$$\begin{aligned} {}_2\omega_1^e(x) &= {}_2\omega_1^e(x; a_1, a_2; b) \equiv {}_2\omega_1^o(x) = {}_2\omega_1^o(x; a_1, a_2; b) \\ &= \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(b)\Gamma(a_1 + a_2 - b - 1)} (1-x)^{a_1+a_2-b-2} {}_2F_1 \left(\begin{matrix} a_1 - b, a_2 - b \\ a_1 + a_2 - b - 1 \end{matrix} \middle| 1-x \right) \end{aligned} \quad (46)$$

If $b = a_1$ or $b = a_2$, we recover the results of the previous example for $p = 1$, $q = 0$.

For any positive parameters p, q , the weight functions solving the moment problems (34) and (35) are provided by the expression

$$\begin{aligned} {}_p\omega_q^e(x) &= {}_p\omega_q^e(x; a_1, \dots, a_p; b_1, \dots, b_q) \equiv {}_p\omega_q^o(x) = {}_p\omega_q^o(x; a_1, \dots, a_p; b_1, \dots, b_q) \\ &= \frac{\Gamma(a_1) \cdots \Gamma(a_p)}{\Gamma(b_1) \cdots \Gamma(b_p)} G_{p,q+1}^{q+1,0} \left(x \middle| \begin{matrix} a_1 - 1, \dots, a_p - 1 \\ b_1 - 1, \dots, b_q - 1, 0 \end{matrix} \right). \end{aligned} \quad (47)$$

Provided these results, we readily obtain the completeness relations corresponding to the families of the states $|p, q, z\rangle_e$ and $|p, q, z\rangle_o$.

IV. PHOTON-COUNTING STATISTICS

In this section, we compute the expectation value of $(\hat{a}^\dagger)^s \hat{a}^r$ in the generalized coherent states $|p, q, z\rangle_e$ and $|p, q, z\rangle_o$ and deduce the corresponding mandel parameter.

From the actions of the operators \hat{a} and \hat{a}^\dagger on the state $|n\rangle$ in (7) we deduce that

$$\hat{a}^r |n\rangle = \sqrt{\frac{n!}{(n-r)!}} |n-r\rangle, \quad 0 \leq r \leq n \quad (48)$$

and

$$(\hat{a}^\dagger)^s |n\rangle = \sqrt{\frac{(n+s)!}{n!}} |n+s\rangle. \quad (49)$$

The GHCS defined in (8) have the Fock representation

$$\langle n | p, q, z \rangle = \frac{z^n}{\sqrt{{}_p\rho_q(n) {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; |z|^2)}} \quad (50)$$

from which the photon number distribution follows as:

$$\mathcal{P}_{|p,q,z\rangle}(n) = |\langle n | p, q, z \rangle|^2 = \frac{x^n}{{}_p\rho_q(n) {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x)}, \quad x = |z|^2. \quad (51)$$

Besides, we can prove the following statement.

Proposition 1. *The expectation value of $(\hat{a}^\dagger)^s \hat{a}^r$ in the generalized coherent states $|p, q, z\rangle_e$ and $|p, q, z\rangle_o$ are given by*

$$\langle (\hat{a}^\dagger)^s \hat{a}^r \rangle_e = \bar{z}^s z^r {}_p\mathcal{S}_q^{(s,r)}(|z|^2), \quad s, r = 0, 1, 2, \dots \quad (52)$$

and

$$\langle (\hat{a}^\dagger)^s \hat{a}^r \rangle_o = \bar{z}^s z^r {}_p\tilde{\mathcal{S}}_q^{(s,r)}(|z|^2), \quad s, r = 0, 1, 2, \dots, \quad (53)$$

respectively, where

$${}_p\mathcal{S}_q^{(s,r)}(x) = \frac{1}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \sum_{m=0}^{\infty} \sqrt{\frac{(2m+r)!(2m+s)!}{{}_p\rho_q(2m+s){}_p\rho_q(2m+r)}} \frac{x^{2m}}{(2m)!}, \quad x = |z|^2. \quad (54)$$

and

$${}_p\tilde{\mathcal{S}}_q^{(s,r)}(x) = \frac{1}{{}_pS_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \sum_{m=0}^{\infty} \sqrt{\frac{(2m+r+1)!(2m+s+1)!}{{}_p\rho_q(2m+s+1){}_p\rho_q(2m+r+1)}} \frac{x^{2m+1}}{(2m+1)!}. \quad (55)$$

In particular, for the EGHS, we have

$$\langle (\hat{a}^\dagger)^r \hat{a}^r \rangle_e = \frac{x^r}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \left(\frac{d}{dx} \right)^r {}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x) \quad (56)$$

which is equivalent to

$$\langle (\hat{a}^\dagger)^r \hat{a}^r \rangle_e = x^r \frac{(a_1, \dots, a_p)_r}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \frac{{}_pC_q(a_1+r, \dots, a_p+r; b_1+r, \dots, b_q+r; x)}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \quad (57)$$

if r is even, or

$$\langle (\hat{a}^\dagger)^r \hat{a}^r \rangle_e = x^r \frac{(a_1, \dots, a_p)_r}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \frac{{}_pS_q(a_1+r, \dots, a_p+r; b_1+r, \dots, b_q+r; x)}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \quad (58)$$

if r is odd.

For the OGHS, we have

$$\langle (\hat{a}^\dagger)^r \hat{a}^r \rangle_o = \frac{x^r}{{}_pS_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \left(\frac{d}{dx} \right)^r {}_pS_q(a_1, \dots, a_p; b_1, \dots, b_q; x) \quad (59)$$

which is equivalent to

$$\langle (\hat{a}^\dagger)^r \hat{a}^r \rangle_o = x^r \frac{(a_1, \dots, a_p)_r}{{}_pS_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \frac{{}_pS_q(a_1+r, \dots, a_p+r; b_1+r, \dots, b_q+r; x)}{{}_pS_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \quad (60)$$

if r is even, or

$$\langle (\hat{a}^\dagger)^r \hat{a}^r \rangle_o = x^r \frac{(a_1, \dots, a_p)_r {}_pC_q(a_1 + r, \dots, a_p + r; b_1 + r, \dots, b_q + r; x)}{(b_1, \dots, b_q)_r {}_pS_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \quad (61)$$

if r is odd, where $(a_1, \dots, a_p)_n := (a_1)_n (a_2)_n \dots (a_p)_n$.

Proof. Indeed, for $s, r = 0, 1, 2, \dots$, we have

$$\begin{aligned} \langle (\hat{a}^\dagger)^s \hat{a}^r \rangle_e &= {}_e \langle z, p, q | (\hat{a}^\dagger)^s \hat{a}^r | p, q, z \rangle_e \\ &= \frac{1}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; |z|^2)} \\ &\times \sum_{m=0}^{\infty} \sum_{2n=r}^{\infty} \sqrt{\frac{(2n)! (2n - r + s)!}{{}_p\rho_q(2m) {}_p\rho_q(2n) (2n - r)! (2n - r)!}} \bar{z}^{2m} z^{2n} \langle 2m | 2n + s - r \rangle \\ &= \frac{1}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; |z|^2)} \\ &\times \sum_{2n=r}^{\infty} \sqrt{\frac{(2n)! (2n - r + s)!}{{}_p\rho_q(2n + s - r) {}_p\rho_q(2n) (2n - r)! (2n - r)!}} \bar{z}^{2n+s-r} z^{2n} \\ &= \frac{\bar{z}^s z^r}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; |z|^2)} \sum_{n=0}^{\infty} \sqrt{\frac{(2n + r)! (2n + s)!}{{}_p\rho_q(2n + s) {}_p\rho_q(2n + r) (2n)!}} |z|^{4n}. \quad (62) \end{aligned}$$

In the special case, when $s = r$, we have

$$\begin{aligned} \langle (\hat{a}^\dagger)^r \hat{a}^r \rangle_e &= \frac{x^r}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x^2)} \sum_{n=0}^{\infty} \frac{(2n + r)!}{{}_p\rho_q(2n + r)} \frac{x^{2n}}{(2n)!} \\ &= \frac{x^r}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \sum_{r=2n}^{\infty} \frac{(2n)!}{{}_p\rho_q(2n)} \frac{x^{2n-r}}{(2n - r)!} \\ &= \frac{x^r}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \left(\frac{d}{dx} \right)^r {}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x), \quad (63) \end{aligned}$$

where $x = |z|^2$. In the same way, we compute $\langle (\hat{a}^\dagger)^s \hat{a}^r \rangle_o$, which achieves the proof.

In particular,

1. for $r = 1$, we deduce the expectation value of the number operator as

$$\langle N \rangle_e \equiv \langle \hat{a}^\dagger \hat{a} \rangle_e = x \left[\frac{\prod_{i=1}^p a_i}{\prod_{i=1}^q b_i} \right] \frac{{}_pS_q(a_1 + 1, \dots, a_p + 1; b_1 + 1, \dots, b_q + 1; x)}{{}_pC_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \quad (64)$$

and

$$\langle N \rangle_o \equiv \langle \hat{a}^\dagger \hat{a} \rangle_o = x \left[\frac{\prod_{i=1}^p a_i}{\prod_{i=1}^q b_i} \right] \frac{{}_pC_q(a_1 + 1, \dots, a_p + 1; b_1 + 1, \dots, b_q + 1; x)}{{}_pS_q(a_1, \dots, a_p; b_1, \dots, b_q; x)}, \quad x = |z|^2; \quad (65)$$

2. for $r = 2$, the computation of the expectation value of the square of the number operator $N^2 = (\hat{a}^\dagger)^2 \hat{a}^2 + \hat{a}^\dagger \hat{a}$ turns to be

$$\begin{aligned} \langle N^2 \rangle_e &= x^2 \left[\frac{\prod_{i=1}^p a_i(a_i + 1)}{\prod_{i=1}^q b_i(b_i + 1)} \right] \frac{{}_p C_q(a_1 + 2, \dots, a_p + 2; b_1 + 2, \dots, b_q + 2; x)}{{}_p C_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \\ &+ x \left[\frac{\prod_{i=1}^p a_i}{\prod_{i=1}^q b_i} \right] \frac{{}_p S_q(a_1 + 1, \dots, a_p + 1; b_1 + 1, \dots, b_q + 1; x)}{{}_p C_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \end{aligned} \quad (66)$$

and

$$\begin{aligned} \langle N^2 \rangle_o &= x^2 \left[\frac{\prod_{i=1}^p a_i(a_i + 1)}{\prod_{i=1}^q b_i(b_i + 1)} \right] \frac{{}_p S_q(a_1 + 2, \dots, a_p + 2; b_1 + 2, \dots, b_q + 2; x)}{{}_p S_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \\ &+ x \left[\frac{\prod_{i=1}^p a_i}{\prod_{i=1}^q b_i} \right] \frac{{}_p C_q(a_1 + 1, \dots, a_p + 1; b_1 + 1, \dots, b_q + 1; x)}{{}_p S_q(a_1, \dots, a_p; b_1, \dots, b_q; x)}, \quad x = |z|^2. \end{aligned} \quad (67)$$

Commonly, the photon-counting statistics of the coherent states is investigated by evaluating the Mandel parameter defined as [28]

$$Q_i := \frac{\langle N^2 \rangle_i - \langle N \rangle_i^2}{\langle N \rangle_i} - 1, \quad (68)$$

where

$$\langle N \rangle_i = {}_i \langle z | N | z \rangle_i \quad (69)$$

The coherent state for which $Q_i = 0$, $Q_i < 0$ and $Q_i > 0$ corresponds to Poissonian, sub-Poissonian (non-classical) and super-Poissonian state [29–31], respectively .

Since the EGHS and the OGHS are orthogonal, we get

$$\langle \hat{a} \rangle_e = \langle \hat{a} \rangle_o = 0 = \langle \hat{a}^\dagger \rangle_e = \langle \hat{a}^\dagger \rangle_o \quad (70)$$

where

$$\langle \hat{X} \rangle_{e(o)} = {}_{e(o)} \langle p, q, z | \hat{X} | p, q, z \rangle_{e(o)}.$$

The Mandel parameter (68) is reduced to

$$\begin{aligned} Q_e &= x \left\{ \frac{\prod_{i=1}^p (a_i + 1)}{\prod_{i=1}^q (b_i + 1)} \frac{{}_p C_q(a_1 + 2, \dots, a_p + 2; b_1 + 2, \dots, b_q + 2; x)}{{}_p S_q(a_1 + 1, \dots, a_p + 1; b_1 + 1, \dots, b_q + 1; x)} \right. \\ &\quad \left. - \frac{\prod_{i=1}^p a_i}{{}_p C_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \frac{{}_p S_q(a_1 + 1, \dots, a_p + 1; b_1 + 1, \dots, b_q + 1; x)}{{}_p C_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \right\} \end{aligned} \quad (71)$$

and

$$Q_o = x \left\{ \frac{\prod_{i=1}^p (a_i + 1)}{\prod_{i=1}^q (b_i + 1)} \frac{{}_p S_q(a_1 + 2, \dots, a_p + 2; b_1 + 2, \dots, b_q + 2; x)}{{}_p C_q(a_1 + 1, \dots, a_p + 1; b_1 + 1, \dots, b_q + 1; x)} \right.$$

$$- \frac{\prod_{i=1}^p a_i {}_p C_q(a_1 + 1, \dots, a_p + 1; b_1 + 1, \dots, b_q + 1; x)}{\prod_{i=1}^q b_i {}_p S_q(a_1, \dots, a_p; b_1, \dots, b_q; x)} \Big\}. \quad (72)$$

The probability of finding n quanta in the EGHS and OGHS are given by

$${}_p \mathcal{P}_q^{even}(x, n) := |\langle 2n | p, q, z \rangle_e|^2 = \frac{x^{2n}}{{}_p \rho_q(2n) {}_p C_q(a_1, \dots, a_p; b_1, \dots, b_q; x)}, \quad (73)$$

and

$${}_p \mathcal{P}_q^{odd}(x, n) := |\langle 2n + 1 | p, q, z \rangle_o|^2 = \frac{x^{2n+1}}{{}_p \rho_q(2n + 1) {}_p S_q(a_1, \dots, a_p; b_1, \dots, b_q; x)}, \quad x = |z|^2, \quad (74)$$

respectively. In particular when $p = 0 = q$, we have

$${}_0 \mathcal{P}_0^{even}(x, n) := |\langle 2n | -, -, z \rangle_e|^2 = \frac{x^{2n}}{(2n)! \cosh x}, \quad (75)$$

and

$${}_0 \mathcal{P}_0^{odd}(x, n) := |\langle 2n + 1 | -, -, z \rangle_o|^2 = \frac{x^{2n+1}}{(2n + 1)! \sinh x}, \quad x = |z|^2, \quad (76)$$

V. QUANTUM OPTICAL PROPERTIES AND GEOMETRY OF THE GHS

$|p, q, z\rangle_i$ **FOR** $p = 1, q = 0$

Now, we exploit the results issued from the above section to derive some quantum optical properties and describe the geometry of the states $|p, q, z\rangle_e$ and $|p, q, z\rangle_o$ for $p = 1, q = 0$.

A. Quantum optical properties

The expectation value of $(\hat{a}^\dagger)^s \hat{a}^r$ in the coherent states $|1, 0, z\rangle_e = |a, -, z\rangle_e$ and $|1, 0, z\rangle_o = |a, -, z\rangle_o$ are given by

$$\langle (\hat{a}^\dagger)^s \hat{a}^r \rangle_e = \bar{z}^s z^r {}_1 \mathcal{S}_0(s, r)(|z|^2), \quad s, r = 0, 1, 2, \dots \quad (77)$$

and

$$\langle (\hat{a}^\dagger)^s \hat{a}^r \rangle_o = \bar{z}^s z^r {}_1 \tilde{\mathcal{S}}_0^{(s, r)}(|z|^2), \quad s, r = 0, 1, 2, \dots, \quad (78)$$

respectively. In particular,

$$\langle (\hat{a}^\dagger)^r \hat{a}^r \rangle_e = \frac{x^r}{{}_1 C_0(a; -, x)} \left(\frac{d}{dx} \right)^r {}_1 C_0(a; -, x) = \begin{cases} x^r(a)_r \frac{{}_1 C_0(a+r; -, x)}{{}_1 C_0(a; -, x)} & \text{if } r \text{ even} \\ x^r(a)_r \frac{{}_1 S_0(a+r; -, x)}{{}_1 C_0(a; -, x)} & \text{if } r \text{ odd} \end{cases} \quad (79)$$

and

$$\langle (\hat{a}^\dagger)^r \hat{a}^r \rangle_o = \frac{x^r}{{}_1S_0(a; -, x)} \left(\frac{d}{dx} \right)^r {}_1S_0(a; -, x) = \begin{cases} x^r(a)_r \frac{{}_1S_0(a+r; -, x)}{{}_1S_0(a; -, x)} & \text{if } r \text{ even} \\ x^r(a)_r \frac{{}_1C_0(a+r; -, x)}{{}_1S_0(a; -, x)} & \text{if } r \text{ odd,} \end{cases} \quad (80)$$

where $x = |z|^2$.

For $r = 1$, we deduce the expectation value of the number operator as

$$\langle N \rangle_e \equiv \langle \hat{a}^\dagger \hat{a} \rangle_e = ax \frac{{}_1S_0(a+1; -, x)}{{}_1C_0(a; -, x)} \quad (81)$$

and

$$\langle N \rangle_o \equiv \langle \hat{a}^\dagger \hat{a} \rangle_o = ax \frac{{}_1C_0(a+1; -, x)}{{}_1S_0(a; -, x)}, \quad x = |z|^2. \quad (82)$$

By using the expectation value of the operator $N^2 = (\hat{a}^\dagger)^2 \hat{a}^2 + N$ provided by

$$\langle N^2 \rangle_e = a(a+1)x^2 \frac{{}_1C_0(a+2; -, x)}{{}_1C_0(a; -, x)} + ax \frac{{}_1S_0(a+1; -, x)}{{}_1C_0(a; -, x)} \quad (83)$$

and

$$\langle N^2 \rangle_o = a(a+1)x^2 \frac{{}_1S_0(a+2; -, x)}{{}_1S_0(a; -, x)} + ax \frac{{}_1C_0(a+1; -, x)}{{}_1S_0(a; -, x)} \quad (84)$$

one readily finds

$$Q_e(x) = x \left((a+1) \frac{{}_1C_0(a+2; -, x)}{{}_1S_0(a+1; -, x)} - a \frac{{}_1S_0(a+1; -, x)}{{}_1C_0(a; -, x)} \right), \quad (85)$$

and

$$Q_o(x) = x \left((a+1) \frac{{}_1S_0(a+2; -, x)}{{}_1C_0(a+1; -, x)} - a \frac{{}_1C_0(a+1; -, x)}{{}_1S_0(a; -, x)} \right), \quad x = |z|^2. \quad (86)$$

For $x \ll 1$, the Mandel parameter Q_e is reduced to

$$Q_e(x) = 1 + o(x^2). \quad (87)$$

Therefore, the EGHS correspond to the super-Poissonian states. Contrary to that,

$$Q_o(x) = -1 + o(x^2) < 0 \quad (88)$$

indicating that the OGHS correspond to the sub-Poissonian states. The probability of finding n quanta in the EGHS and OGHS are given by

$${}_1\mathcal{P}_0^{even}(x, n) := |\langle 2n | a, -, z \rangle_e|^2 = \frac{\left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n}{n! \left(\frac{1}{2}\right)_n} \frac{x^{2n}}{{}_2F_1 \left(\begin{matrix} \frac{a}{2}, \frac{a+1}{2} \\ \frac{1}{2} \end{matrix} \middle| x^2 \right)}, \quad (89)$$

and

$${}_1\mathcal{P}_0^{odd}(x, n) := |\langle 2n+1|a, -, z\rangle_o|^2 = \frac{\left(\frac{a+1}{2}\right)_n \left(\frac{a+2}{2}\right)_n}{n! \left(\frac{3}{2}\right)_n} \frac{x^{2n}}{{}_2F_1\left(\frac{a+1}{2}, \frac{a+2}{2} \middle| x^2\right)}, \quad x = |z|^2, \quad (90)$$

respectively.

B. The statistical distribution

The thermodynamics properties are shown to be determined by the partition function Z defined by

$$Z = Tr(e^{-\beta H}) = \sum_{n=0}^{\infty} \langle n|e^{-\beta H}|n\rangle, \quad (91)$$

where $\beta = 1/kT$. We assume the Hamiltonian to be defined as follows:

$$H := \omega \hat{a}^\dagger \hat{a}. \quad (92)$$

Now we can compute the partition function for the generalized oscillator as follows:

$$Z = \sum_{n=0}^{\infty} \langle n|e^{-\beta H}|n\rangle = \frac{1}{1 - e^{-\beta \omega}}. \quad (93)$$

The statistical distribution of the operator \hat{O} is defined through the formula

$$\langle \hat{O} \rangle := \frac{1}{Z} Tr(e^{-\beta H} \hat{O}). \quad (94)$$

Using (48) and (49), the mean value of the product of operators $(\hat{a}^\dagger)^r (\hat{a})^r$ is given by:

$$\langle (\hat{a}^\dagger)^r (\hat{a})^r \rangle = \frac{(1 - e^{-\beta \omega})}{(-\omega)^r} \left(\frac{d}{d\beta} \right)^r (1 - e^{-\beta \omega})^{-1}. \quad (95)$$

For $r = 1$, we recover, as expected, the well known Green function $\langle \hat{a}^\dagger \hat{a} \rangle$ which is the mean occupation number, i.e.,

$$\langle \hat{a}^\dagger \hat{a} \rangle = \frac{1}{e^{\omega/kT} - 1}. \quad (96)$$

C. Geometry of the states $|a, -, z\rangle_e$ and $|a, -, z\rangle_o$

The geometry of a quantum state space can be described by the corresponding metric tensor. This real and positive definite metric is defined on the underlying manifold that

the quantum states form, or belong to, by calculating the distance function (line element) between two quantum states. It is also known as a Fubini-Study metric of the ray space. The knowledge of the quantum metric enables to calculate quantum mechanical transition probability and uncertainties [33, 34]. The map $z \mapsto |a, -, z\rangle_i$, $i = e, o$, defines a map from the space C of complex numbers onto a continuous subset of unit vectors in Hilbert space and generates in the latter a two-dimensional surface with the following Fubini-Study metric:

$$d\sigma^2 := ||d|a, -, z\rangle_i||^2 - |{}_i\langle z, -, a|d|1, -, z\rangle_i|^2. \quad (97)$$

Proposition 2. *The Fubini-Study metric (97) is reduced to*

$$d\sigma^2 = {}_1W_0^i(x)d\bar{z}dz, \quad x = |z|^2, \quad (98)$$

where

$${}_1W_0^{even}(x) = a \frac{d}{dx} \left[x \frac{{}_1S_0(a+1; -; x)}{{}_1C_0(a; -; x)} \right] = \frac{d}{dx} \langle N \rangle_e \quad (99)$$

and

$${}_1W_0^{odd}(x) = a \frac{d}{dx} \left[x \frac{{}_1C_0(a+1; -; x)}{{}_1S_0(a; -; x)} \right] = \frac{d}{dx} \langle N \rangle_o. \quad (100)$$

Proof. Computing $d|a, -, z\rangle_i$, $i = e, o$, by taking into account the fact that any change of the form $d|a, -, z\rangle_i = \alpha|a, -, z\rangle_i$, $\alpha \in C$, has zero distance, we get

$$d|a, -, z\rangle_e = \frac{1}{\sqrt{{}_1C_0(a; -; |z|^2)}} \sum_{n=0}^{\infty} \frac{2nz^{2n-1}}{\sqrt{{}_1\rho_0(2n)}} |2n\rangle dz. \quad (101)$$

Then,

$$\begin{aligned} ||d|a, -, z\rangle_e||^2 &= \frac{1}{{}_1C_0(a; -; |z|^2)} \sum_{n=0}^{\infty} \frac{(2n)^2 |z|^{2(2n-1)}}{{}_1\rho_0(2n)} d\bar{z}dz \\ &= \frac{1}{{}_1C_0(a; -; |z|^2)} \left(\sum_{n=0}^{\infty} \frac{2n|z|^{2(2n-1)}}{{}_1\rho_0(2n)} + |z|^2 \sum_{n=0}^{\infty} \frac{2n(2n-1)|z|^{2(2n-2)}}{{}_1\rho_0(2n)} \right) d\bar{z}dz \\ &= \frac{a {}_1S_0(a+1; -; x) + a(a+1)x {}_1C_0(a+2; -; x)}{{}_1C_0(a; -; x)} d\bar{z}dz \\ &= a \frac{[x {}_1S_0(a+1; -; x)]'}{{}_1C_0(a; -; x)} d\bar{z}dz \end{aligned} \quad (102)$$

and

$$|{}_e\langle z, -, a|d|a, -, z\rangle_e|^2 = \left| \frac{1}{{}_1C_0(a; -; x)} \sum_{n=0}^{\infty} \frac{2n|z|^{2(2n-1)}}{{}_1\rho_0(2n)} \bar{z}dz \right|^2$$

$$= a^2 x \left[\frac{{}_1S_0(a+1; -; x)}{{}_1C_0(a; -; x)} \right]^2 d\bar{z}dz. \quad (103)$$

Therefore,

$$\begin{aligned} d\sigma^2 &= \left(a \frac{(x {}_1S_0(a+1; -; x))'}{{}_1C_0(a; -; x)} - a^2 x \frac{{}_1S_0^2(a+1; -; x)}{{}_1C_0^2(a; -; x)} \right) d\bar{z}dz \\ &= \frac{d}{dx} \left[ax \frac{{}_1S_0(a+1; -; x)}{{}_1C_0(a; -; x)} \right] d\bar{z}dz \\ &= \left(\frac{d}{dx} \langle N \rangle_e \right) d\bar{z}dz, \end{aligned} \quad (104)$$

where $(\cdot)'$ denotes the derivative with respect to x .

In the same way, for the OGHS

$$d\sigma^2 = \left(\frac{d}{dx} \langle N \rangle_o \right) d\bar{z}dz, \quad (105)$$

which achieves the proof.

For $x \ll 1$, we have

$${}_1W_0^{even}(x) = 2a(a+1)x + o(x^2), \quad {}_1W_0^{odd}(x) = a(a+1)x + o(x^2). \quad (106)$$

Conclusion

In this work, we have investigated a large class of generalized hypergeometric states $|p, q, z\rangle$, depending on a complex variable z and two sets of parameters, (a_1, \dots, a_p) and (b_1, \dots, b_q) . Besides, we have defined and analyzed even and odd generalized hypergeometric states $|p, q, z\rangle_e$ and $|p, q, z\rangle_o$. We have solved the moment problem using the Mellin transform techniques. For particular values of p and q , we have discussed the photon-counting statistics, quantum optical properties and geometry of these states.

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